thus drawing a much weaker dependance on the velocity ratio than that used by Durando in his Eq. (8). Hence, the basic form assumed for the spread of the vortices does not conform with the observed variations in vortex centre position and unfortunately, any modification to the form of Eq. (8) will be unrealistic since the self-consistency of the solution will be destroyed.

Conclusions

The proposed vortex model of Durando does not appear to be a satisfactory representation of the vortex dominated region of a deflected jet. In view of the success of the model when applied to separated flows on slender bodies this is disappointing. The failure is thought to be due to the very different vorticity distributions in the two flowfields. In the separated flows, well-defined vortex sheets feed the main vortices which develop with relatively compact cores. Thus, the mathematical analogy is closely representative of the real situation especially as the shape of the feeding vortex sheet does not have to be specified and can be curved as in the physical flow.

The distribution of vorticity in the jet plume is very different; the vortices being far more diffuse to the extent that the decay in circulation in the streamwise direction appears to be a direct consequence of the progressive overlapping of the two vorticity fields. This difference leads to the failure of Durando's theory on two counts. Firstly his model based on two concentrated cores is inadequate from the physical viewpoint of being unrepresentative of the real flow. Secondly, because of the cancellation of vorticity by overlapping, the circulation is far lower and changes in a manner quite different to that predicted by the criteria he uses. Unfortunately incorporating the observed behaviour of the vortices into his analysis destroys the self-consistency of the solution.

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Reply by Author to A. M. Thompson

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In his Comment, A. M. Thompson[1] shows large numerical discrepancies between his data and the predictions of the jet vortex model proposed by the author in Ref. 2. There is no question that Eq. (14) of Ref. 2 will greatly overpredict the strength of the counter-rotating vortices in the jet plume. A great deal of the discrepancy has been traced to errors made in adjusting the empirical constants of Ref. 2 to the jet plume characteristics given by Pratte and Baines,[3] and corrected values for these constant will be given below. Although satisfactory quantitative agreement is still not achieved, the author does not feel that Thompson's data as given in Ref. 1 may be used to conclusively prove or disprove the basic validity of his model. As stated in Ref. 2, the model is restricted to the "vortex zone" of the jet. As defined by Pratte and Baines,[3] this zone lies in the range

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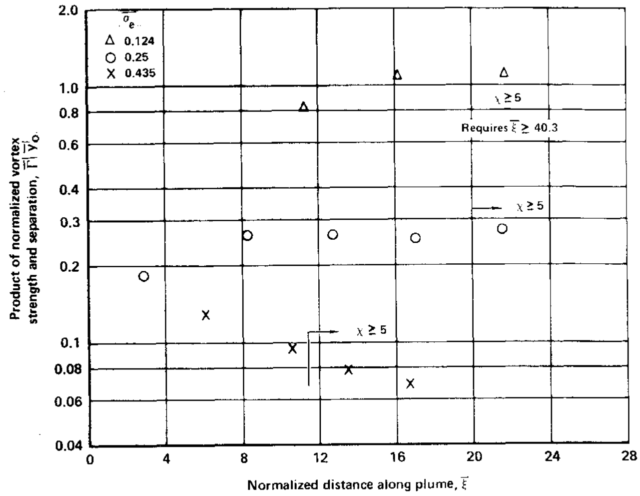

Fig. 1 Experimental values of the product of vortex strength and separation.

$$\chi > 5$$

where

$$\chi = \sigma_e \xi / d_e$$

Examination of Thompson's Figs. 1 and 2 reveals that only three of his data points fall in this range. In addition, it has been shown by Keffer and Baines[4] that for values of σ_e greater than approximately 0.25 (their parameter $R = 1/\sigma_e$). the jet plume is affected by the presence of the wall from which it emerges. Thompson's data at $\sigma_e = 0.435$ may therefore show the influence of this effect. As the range of validity of the model is approached, some of the trends exhibited by Thompson's data are not inconsistent with the conclusions of Ref. 2. In discussing these trends, conditions at fixed values of σ_e will be treated separately from the questions of correlations for different values of σ_e.

Conditions at a Fixed Value of σ_e

Equations (5) and (6) of Ref. 2 may be rewritten as follows:

$$\bar{\Gamma} = \bar{K}'/\bar{y}_o \quad (1)$$

$$d\bar{z}_o/d\bar{\xi} = \bar{\Gamma}/y_o \quad (2)$$

where

$$\bar{\xi} = \xi/d_e, \quad \bar{y} = y/d_e, \quad \bar{\Gamma} = \Gamma/4\pi U_\infty d_e, \quad \bar{K}'(\sigma_e) = K'/4\pi U_\infty d_e^2$$

Equation (1) is a direct result of the basic assumption that the net force on the vortices and their connecting sheet is zero. The product $\bar{\Gamma}\bar{y}_o$, calculated from Thompson's data,[1] is plotted in Fig. 1. Values of $\bar{\xi}$ corresponding to $\chi = 5$ are also indicated in the figure. For $\sigma_e = 0.124$, the greatest value of $\bar{\xi}$ calculated from Fig. 1 of Ref. 1, and that calculated from Fig. 2 of Ref. 1 disagreed by somewhat more than two jet diameters. Since this exceeded the discrepancies which might be expected from inaccuracies in reading the graphs, this point has not been included. Unfortunately, all values of $\bar{\xi}$ for $\sigma_e = 0.124$ lie well below that corresponding to the beginning of the vortex zone. For $\sigma_e = 0.25$, the product $\bar{\Gamma}\bar{y}_o$ is fairly constant, even for values of $\bar{\xi}$ below that corresponding to $\chi = 5$. This trend appears to support Eq. (1), and thus the basic assumption of Ref. 2.

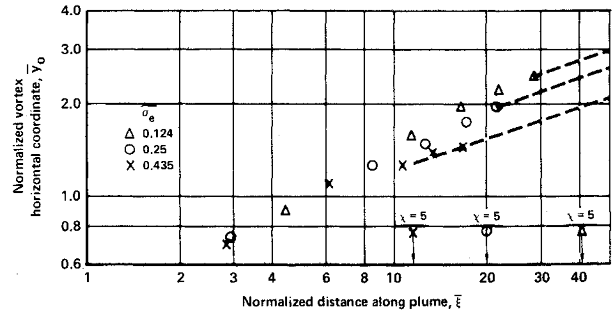

Fig. 2 Distribution of vortex spacing along the jet trajectory.

For $\sigma_e = 0.435$, the product $\bar{\Gamma}\bar{y}_o$ exhibits a very different behavior. It decreases with increasing $\bar{\xi}$, instead of increasing to a constant value as for $\sigma_e = 0.25$. This trend may be in contradiction to the basic assumption of Ref. 2, or it may be caused by the influence of the wall from which the jet emerges, at this relatively low value of jet velocity.

The author[2] has assumed that the spacing between vortices varies as $\chi^{1/3}$, in the same way as the jet plume width measured by Pratte and Baines. At a fixed value of σ_e, this implies

$$\bar{y}_o = \bar{y}_v(\sigma_e)\bar{\xi}^{1/3} \tag{3}$$

Figure 2 shows the distribution of $\bar{y}_o$ along the jet plume, as calculated from Thompson's Fig. 2. The lower limit of the vortex zone for each value of σ_e is also indicated in the figure. The dotted lines have a slope of $\frac{1}{3}$ and have been adjusted to go through the last data point at each value of σ_e. They have been drawn to indicate that a $\frac{1}{3}$-slope straight line is not an unreasonable extrapolation to Thompson's data. Once again, however, there is almost no overlap between the range of ξ for which data was obtained, and the range of applicability of the author's model.

Conditions at Different Values of σ_e

Questions of data correlation in the range of validity of the model must be based upon the three data points for $\chi \geqq 5$. Furthermore, the two points for $\sigma_e = 0.435$ may well be misleading because of wall effects. Thus, it does not appear possible to draw conclusions with any certainty. There appears to be some confusion about how the form of similarity variables which correlate data for different values of σ_e will affect the validity of the model. This question will be discussed briefly.

Thompson states that because his vortex spacing measurements correlate as

$$\bar{y}_o\sigma_e^{(0.31)} = f[\bar{\xi}\sigma_e^{(0.31)}]$$

instead of

$$\bar{y}_o\sigma_e = f(\sigma_e\bar{\xi})$$

as assumed in Ref. 2, the self-consistency of the model is destroyed. This is not the case at all. The question of self-consistency cannot be discussed without including the specific form of the function f, as well as the relationship between vortex spacing and vortex trajectory. This may be shown by combining Eqs. (1) and (2) to obtain

$$d\bar{z}_o/d\bar{\xi} = \bar{K}'/y_o^2 \tag{4}$$

Now, suppose that vortex spacing and trajectory correlate in the general form

$$\sigma_e^m\bar{y}_o = Y_o(\sigma_e^n\bar{\xi}) \tag{5}$$

$$\sigma_e^p\bar{z}_o = Z_o(\sigma_e^q\bar{\xi}) \tag{6}$$

where m, n, p, and q are constants.

In terms of these similarity variables, Eq. (4) then becomes

$$d[Z_o(\sigma_e^q\bar{\xi})]/d(\sigma_e^q\bar{\xi}) = \{\bar{K}'\sigma_e^{(2m-q+p)}\}/Y_o^2(\sigma_e^n\bar{\xi}) \tag{7}$$

In general, Eq. (7) would require for consistency

$$n = q \tag{8}$$

$$\bar{K}'\sigma_e^{2m-q+p} = K \tag{9}$$

where K is a constant independent of σ_e. Substituting Eqs. (5) and (9) into Eq. (1)

$$\Gamma^*(\sigma_e^q\bar{\xi}) = \bar{\Gamma}\sigma_e^{(p-q+m)} = K/Y_o(\sigma_e^n\bar{\xi}) \tag{10}$$

where Γ^* is independent of σ_e. Condition (8) thus requires that vortex spacing and trajectory correlate in terms of similarity variables multiplied by the same power of σ_e. However, if the function Y_o is some power of the variable $\sigma_e^n\bar{\xi}$, even this condition may be relaxed, for if

$$\sigma_e^m\bar{y}_o = Y_v(\sigma_e^n\bar{\xi})^\beta$$

then Eq. (7) may be written as

$$d[Z_o(\sigma_e^q\bar{\xi})]/d(\sigma_e^q\bar{\xi}) = \{\bar{K}'\sigma_e^{[2m-q+p+2\beta(q-n)]}\}/(\sigma_e^q\bar{\xi})^{2\beta} \tag{11}$$

If for a given β, integration of Eq. (11) leads to the correct jet trajectory, the model is self-consistent regardless of the specific values of m, n, p, and q. The variables used in Ref. 2 correspond to

$$m = n = p = q = 1$$

$$\beta = \tfrac{1}{3}$$

Figure 3 shows the data of Ref. 1 plotted in the similarity variables of Ref. 3. The plume half-width, from Fig. 9 of Ref. 3, is also shown. There is no doubt that for the range in which the data was taken, Thompson's variables[1] provide a much better correlation than that shown in Fig. 3. However, it is not clear that this will be so for $\chi \geqq 5$, since not enough points have been obtained in this range.

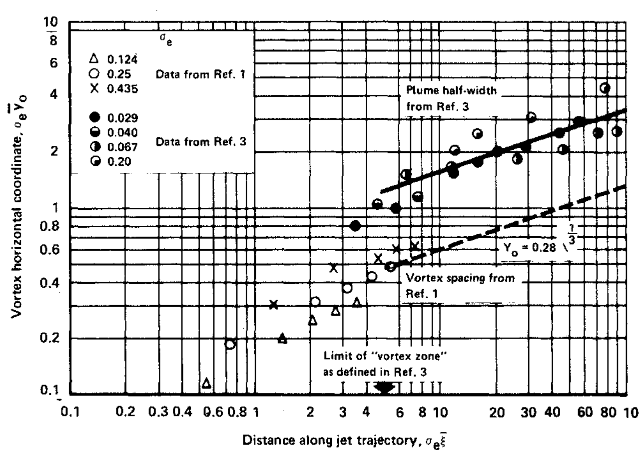

Fig. 3 Vortex separation and plume half-width in similarity variables.

Corrected Values of Empirical Constants

The numerical value chosen for Y_v in Ref. 2 is incorrect. It was obtained by matching Eq. (8) of Ref. 2 directly to Fig. 9 of Ref. 3. This figure, however, shows the entire plume width, whereas the vortex location Y_o should really be some fraction of the plume half-width. Eq. (12) of Ref. 2 may be rewritten in the form

$$\Gamma^* = Z_v Y_v/3\chi^{1/3} \tag{12}$$

where

$$Z_v = 3K/Y_v^2$$

A corrected value for Y_v has been estimated from Fig. 3. This is

$$Y_v = 0.28$$

with

$$Z_v = 1.63 \tag{13}$$

as in Ref. 2, the new expression for Γ^* would be

$$\Gamma^* = 0.15/\chi^{1/3} \quad (14)$$

While this new value of Y_v substantially decreases the predicted vortex strength, Eq. (14) still overestimates the vortex strength by a factor of more than two. One further source of error may lie in the value chosen for Z_v. This has been taken from Fig. 5 of Ref. 3, which implies that the vortices have the same Z-coordinate as the midpoint of the jet plume. The vortices might actually lie below the line, which would require a smaller value of Z_v, and would further reduce the numerical constant in Eq. (14).

Conclusions

It appears that Thompson's data cannot be used to conclusively prove or disprove the validity of the author's model, because most of it lies outside the range of applicability of the model. His statement that a form of similarity variables which differs from that used by the author destroys the self-consistency of the model does not seem to be correct. As far as the vortex zone of the jet is concerned, Thompson's data sheds little light on the question of correlations for different values of σ_e due to the scarcity of points in this region, and to the possibility of wall effects influencing some of his data. Although some errors in the empirical constants of Ref. 2 have been corrected, the author's model still overpredicts the vortex strength by a factor of more than two. Further adjustment of empirical constants would require additional experimental information, such as vortex spacing and trajectory within the vortex zone of the jet plume.

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Comment on "A New Integral Calculation of Skin Friction on a Porous Plate"

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THE technique[1] of using a double-integral variation of the Kármán-Pohlhausen method was published by Whitehead[2] in 1949. The technique was applied to the case of the laminar boundary layer on an impervious surface and in a pressure gradient.

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Reply by Author to P. S. Granville

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THE author wishes to thank P. S. Granville for his interest and for bringing Whitehead's work[1] to the author's attention.

As Granville correctly pointed out, the idea of using double-integration had indeed appeared earlier in Ref. 1. However, it should be noted that Volkov's technique,[2] which was directly generalized in Ref. 3 to allow for surface mass transfer, was based on a somewhat different use of the idea. Particularly, the determination of skin friction in Ref. 2 differs from that in Ref. 1.

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Comment on "Large Deflection Analysis of Plates"

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IN his recent paper,[1] Yang in common with other workers[2] in the field of large deflections has used Levy's results[3] for a simply-supported square plate under uniform pressure as a standard against which the accuracy of his own finite element results may be assessed. In Ref. 4, a similar but less extensive study than Yang's of the nonlinear behavior of plates, employing conforming triangular finite elements,[5,6] revealed that Levy's original data was not sufficiently accurate for comparison with results from high-precision finite elements; moreover the boundary conditions employed by Levy were not identical to the usual kinematic constraints of a displacement finite element model.

The effect of employing more terms (n) in the Fourier Series expansion of Levy is shown below in Table 1

Table 1 Effect of additional terms on Levy's results

	Membrane $(\sigma/E)(1-\nu^2)\times 10^4$	Bending $(\sigma/E)(1-\nu^2)\times 10^4$	(w/t)
3	0.3865	0.3106	1.82444
5	0.3910	0.3624	1.84061
7	0.3901	0.3385	1.83681
9	0.3904	0.3505	1.83795
11	0.3903	0.3435	1.83750
13	0.3903	0.3478	1.83770
Levy[3]	0.392	0.384	1.846

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